A panorama of numbers [1]

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1 A quick review of history

Prehistoric men were primitive in many respects. They survived mainly on agriculture (in a crude way), hunting and animal husbandry. Their first step towards civilisation was the discovery of languages, which led them to numbers and arithmetic. Counting was the first arithmetic operation discovered/rather developed by man. Numbers were used for counting sheep or counting the fruits in a basket or almost any countable object. Symbols were developed for denoting numbers and a wide variety of tools emerged, from notches made on bones, to knots made on rope, and collections of pebbles. Numbering systems were developed subsequently to denote any arbitrary number using a limited set of symbols. This gave rise to the basic arithmetic operations we use today. So, understanding numbers is essential to understanding arithmetic and eventually, all of mathematics.

1.1 What are numbers ?

This seems like a meaningless, unnecessary question. We need an answer to this question, to put this article in its right perspective. A number is an abstract concept (not an object) used for counting, measuring, labeling. Their are three aspects for each number : its name, its symbolic representation (called a numeral), and the value it represents. Often , these aspects are not distinguished from one another, and are used interchangeably. Since a number is an abstract concept, it cannot be seen or shown all alone. It is not an object, as is usually misunderstood. Ask anyone to show the number five, he/she will instinctively display his/her five fingers. Notice that you are seeing his/her fingers, but NOT the number five. It takes time to accept

[1] This pdf document was created using LaTeX . Texts marked in wine-red colour are click-sensitive hyperlinks.

that numbers (all alone) are invisible. But, numbers are omnipresent and are used everywhere !

The study of (mathematical) properties of numbers, particularly integers, has given rise to a fascinating and rich branch of mathematics – number theory. This itself has given rise to fields like cryptography, algebraic geometry and abstract algebra, and is continuously growing. Many international awards and prizes are awarded even today for exceptional contributions to number theory. Unfortunately, it is also used by promoters of farcical pseudo-sciences like numerology and astrology.

Each language has its own name and symbol for each number. Each of the numbers 1, 2, 3, 4 .... can be a number in itself, or may be a building block of a larger number (in which case, it is called a digit). For example, 259 is a three–digit number, 9 is a single-digit number. The ten decimal digits we see frequently: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are the so called Hindu-Arab numerals (or decimal digits). The system of using decimal digits to create larger numbers, as we knew today, is primarily due to the innovations created by Hindu mathematicians, carried to the Western world by Persian/Arab travellers, resulting in their confused branding as "Hindu-Arab" numerals. In India, we call them as Arabic numerals, and in Arabic countries as Hindu numerals. The "Hindu-Arab" label is a terminological compromise to sound diplomatic.

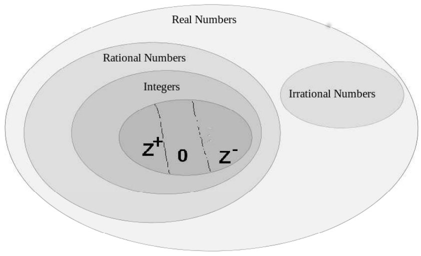
2 A special set of numbers

> Natural numbers were created by God, everything else is the work of men - Kronecker (1823-1891).

The most common form of numbers is a "natural number " [4] . In mathematics, the natural numbers are those used for counting and ordering. In common mathematical terminology, words colloquially used for counting are "cardinal numbers" and words connected to ordering represent "ordinal numbers". However, there are other ways of classifying numbers, using sets.

Integers (aka whole numbers) form a very commonly used set. An integer (from the Latin integer meaning "whole") is colloquially defined as a number that can be written without a fractional component. Peano axioms [1] introduced the concept of zero as a natural number. There is however a lot of debate as to the real origin of "zero"[5].

Integers can thus be partitioned into three disjoint sets : positive integers, zero, negative integers, as $\mathbb{Z} = \mathbb{Z}^+ \cup \{z\} \cup \mathbb{Z}^-$ The set $\mathbb{Z}$ forms part of the set of a larger family of numbers as shown in the figure below :



The symbol $\mathbb{Z}$ derives from the German word Zahl [2], meaning "number" first appeared in Bourbaki's Algèbre (reprinted as Bourbaki 1998, p. 671) [3].

The only numbers that matter are zero, one, and too many.

2.1 Number line

The primitive man started with "natural numbers". As he grew wiser and more daring, he invented "negative" numbers, which was unimaginable in the beginning. So, that led to a number "zero" which was neither postive nor negative, and was yet another much contested concept. Mathematicians invented the "number line", to put all this in perspective and to give a coherent explanation of numbers. Things did not stop with that. The number line gave rise to a "number plane" which we will see soon, in a subsequent section.

A "number line" is a convenient way of visualising $\mathbb{Z}$. A number line is an imaginary (straight) line extending from $-\infty$ to 0 to $+\infty$. Each integer member of $\mathbb{Z}$ is marked as a distinct point on the number line. By convention, the members of $-\infty$ to 0 are placed on the left side of the number line. members of 0 to ∞ are placed on the right side of the number line. 0 comes in the middle. The numbers increase in size monotonically as we proceed from left to right ($-\infty$ to $+\infty$). Inclusion of 0 in the set of natural numbers is still a matter of debate. ∞ is an elusive number which cannot be actually shown on any number line. Some more interesting details of 0 and ∞ are given in [5] and [6].

This same idea can be extended to a number line of real numbers (real line). Of course it is not possible to denote all real numbers on such a real-number line (real line), because there are an infinite number of real numbers between each successive pair of integers. Number lines are used for explaining addition and subtraction, in terms of moving forward and backward on a number line, under the metaphor of a person walking.

3 Some common number sets

Integers have given rise to other families (sets) of numbers .

Symbol	Description of the set	Code
$\mathbb{P}$	prime numbers	`\mathbb{P}`
$\mathbb{Z}$	natural numbers (including zero)	`\mathbb{Z}`
$\mathbb{Z}^+$	positive Integers	`\mathbb{Z}^+`
$\mathbb{Z}^-$	negative Integers	`\mathbb{Z}^-`
$\mathbb{C}$	complex numbers	`\mathbb{C}`
$\mathbb{I}$	imaginary numbers	`\mathbb{I}`
$\mathbb{Q}$	rational numbers	`\mathbb{Q}`
$\mathbb{R}$	real numbers	`\mathbb{R}`
$\mathbb{R}_{>0}$	positive real numbers	`\mathbb{R}_{>0}`
$\mathbb{R}_{\geq0}$	non-negative real numbers	`\mathbb{R}_{\geq0}`
$\{z\}$	zero (a singleton set containing zero only)	
$\{\phi\}$	a null set (empty set)	

Note: Peano's axiom #1 defines the set $\mathbb{Z}$ of natural numbers to include 0 [1].

4 From the number line to the number plane

Instead of a one-dimensioned number line, mathematicians invented a two-dimensioned space to denote a special class of numbers, called "complex" numbers. We use $\mathbb{C}$ to denote the set of complex numbers.

Complex numbers are conveniently represented as points on the complex plane (a 2-dimensioned Euclidean plane), in either of two forms :

- Cartesian coordinates
- Polar coordinates

In the cartesian coordinates system, a complex number c is represented as $c = a + ib$ where $i = \sqrt{-1}$

The concept of a special number called $i = \sqrt{-1}$ may seem insane, but it has led to many useful results and applications in many areas of science. The applications of complex numbers in the real world are endless: control theory, signal analysis, relativity, wave equations, electro-technology (alternating current circuits and devices) and fluid dynamics, all use complex numbers.

The components a and b are called the real part and the imaginary part respectively of c. Complex numbers always come with their two components. Compare this with your shoes, which always come in pairs Each component of c (which is a complex number) is also a number.

In the polar coordinates system, c has a magnitude $M = \sqrt{(a^2 + b^2)}$ and an angle $\phi = tan^{-1}\frac{a}{b}$.

Arithmetic operations usually done on non-complex numbers are also defined for complex numbers. The study of operations on complex numbers, and their properies has given rise to the branch of mathematics called complex analysis.

Moral of the story : *Complex numbers are real !* (pun intended) The qualificative "complex" is just a bad choice for what would better be called as "fanciful" numbers.

5 Closing remarks

The famous German mathematician Carl Friedrich Gauss (17771855) said, "Mathematics is the queen of the sciences, and number theory is the queen of mathematics." But, like many queens, numbers are the least understood objects in science. A study of numbers, their origins, their properties, and their gyneology can lead to a better discovery of mathematics in all its glory. This article is one such attempt.

Readers are requested to send all remarks, criticisms and suggestions to the author at drpartha@gmail.com.

A whole lot of similar articles and tutorial material is available for download from [7] This article is released under a liberal license [8] and published on the web at [9].

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