A crazy mathematical idea called infinity [1]

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GOD was found in infinity – and in zero[1]

1 A crazy mathematical concept

The Indian mathematics prodigy Ramanujan once asked his teacher "if there are zero mangos distributed evenly to zero children, will every child get one mango ?" . The teacher, who did not understand the profoundness of this question, fumbled and mumbled in a way no ten-year old would understand. There is nothing surprising in this story, since, even now, many of us do not know the answer, nor will we ever know fully. This problem has a twin brother, what is x (any x) divided by zero ? The answer is a devil called "infinity" (symbolically represented by ∞). And if that devil is multiplied by zero , will we get zero $[x/y] * 0 = 0$? or $y * [x]/y = x$? or something else ?

It is also not possible to say what $\infty - \infty$ or $\frac{\infty}{\infty}$ would be (because all ∞ are not born equal) . Since all infinity are not equal, there must be one which is the biggest. Right ? Wrong ?

Wrong , thanks to the German mathematician Georg Cantor, who first stated and proved an important theorem, at the end of the 19th century.

Cantor's theorem: The theorem [9] states that the cardinality (numerical size) of a set is strictly less than the cardinality of its power set, or collection of subsets.

[1] Texts marked in wine-red colour are click-sensitive hyperlinks

For any set A , the set of all subsets of A (the power set of A), denoted by $\mathcal{P}(A)$, has a strictly greater cardinality than A itself. Cantor's theorem had immediate and important consequences for the philosophy of mathematics. For instance, by iteratively taking the power set of an infinite set (e.g. cardinals) and applying Cantor's theorem repeatedly, we obtain an endless hierarchy of infinite cardinals, each strictly larger than the one before it. Consequently, Cantor's theorem implies that there is no largest cardinal number (colloquially, *there's no largest infinity*).

There are many more challenges coming our way.

First, let us contemplate on what exactly infinity is. Some say, infinity is a number. Let us imagine we list out all prime numbers:

1, 3, 5, 7, 11, 13, ...

This list will never end (courtesy Euclid). The last term in the above sequence is conveniently called infinity or ∞ . We treat ∞ as a number here.

Thanks to Peano's axioms, we can extend the above line to the other side of 1, and eventually reach $-\infty$.

On the other hand, according to Wikipedia, infinity refers to something that is boundless or endless, or something that is larger than any real number. If it is a number, it must fit somewhere on the mythical number line (where all numbers can be placed as discrete points). The Wikipedia definition above implies that there is no place for infinity on any number line since it is bigger than any number. By contraast, zero has a fixed spot on the number line. Infinity is in fact, related to sets, or rather sizes of sets. Infinity turns out to be a mathematical zombie.

This is the beginning of the many enigmas which haunt infinity.

An interesting list of the queer properties of infinity can be found in [4] and [3]. This article is a synopsis of some properties of infinity.

2 Root of all Paradoxes

Infinity is a weird concept and is the root of many challenges which break our common sense (commonsense). Yet, it plays a very significant role in various branches of mathematics, particularly calculus, operations research, statistics, data science, infinite series, etc. Formally, an infinite series is not a sum but actually what is known as a limit. The renowned master of mathematics , Leonhard Euler, studied infinity and gave us some amazing results for understanding infinity [7]

Zeno's paradoxes: which asserts that when one attempts to travels from 0 to 1 , it always involves travelling half the distance in between. We can go on like this forever, and conclude that the person will never reach the destination. Or, more precisely, will take an infinite amount of time to reach there.

This paradox has many variants and names, collectively called as Zeno's paradoxes [6]

Hilbert's paradoxical grand hotel: where one assumes a hypothetical hotel without any vacant room but still can accommodate infinitely many new guests.

This paradox has many different extensions, each with its own solution and arguments [10]. Raymond Smullyan gives a long explanation of this paradox in his typical story-telling style [4].

Ramanujan's summation : This mathematical challenge was popularised by the indian mathematical prodigy, Ramanujan. Imagine the innocent looking series given below (a series of dots ...... indicates continuation upto infinity):

$$1 + 2 + 3 + 4 + \dots\dots\dots = ? \tag{1}$$

This divergent infinite series is a major callenge to our common sense. The LHS of the above equation :

1. Has only integers. (i.e. has no fractions)
2. All terms are positive
3. The terms are monotonically increasing in size, and go up to infinity

The Indian mathematical genius Ramanujan [8] computed the sum (or rather, the limit) to be an incredible negative fraction : - 1/12

3 Challenges to our common sense

We list some more challenges which we get from ∞ .

You can say goodbye to your intuition and common sense when dealing with infinity. Here is some more reason to hate your intuitions:

- What would be the answer for this equation ?

$$\frac{1+1+1+....}{2+2+2+.....} = ? \tag{2}$$

There are many ways to look at this equation, each of which will lead you to a different answer.

That is why such an expression is called as an indeterminate form [11] and is considered illegitimate.

A few common indeterminate forms (one of which, foxed Ramanujan's maths teacher) are given below:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 * \infty, \infty - \infty, 0^0, 1^0, \infty^0$$

Stay away from such deceptive monsters !

For instance :

$$\frac{1+1+1+....}{2+2+2+.....} =? \tag{3}$$

$$= \frac{\infty}{\infty} \tag{4}$$

$$= 1 \tag{5}$$

Or, look at it this way ::

$$\frac{1+1+1+....}{2+2+2+.....} =? \tag{6}$$

$$= \frac{1+1+1+1+.....}{2(1+1+1+1....)} \tag{7}$$

$$= \frac{1}{2} \tag{8}$$

Or, in yet another (illegitimate) way :

$$\frac{1+1+1+....}{2+2+2+.....} =? \tag{9}$$

$$= \frac{\infty}{\infty} \tag{10}$$

Is this enough to go mad with infinity ? So, be very very careful when you deal with infinity.

4 Closing remarks

Zero and its twin brother, infinity, have a long heritage of several centuries. With some training and imagination, one can visualise zero, but not infinity. It is not easy to explain infinity in a finite time and using a finite space. Nevertheless, the author has taken up this seemingly impossible task. The reader is requested to send all remarks, criticisms and suggestions to the author at drpartha@gmail.com.

A whole lot of similar articles and tutorial material is available for download from [12] This article is released under a liberal license [13] and published on the web at [14].

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