Bridges of Köenigsburg and Euler's Palace

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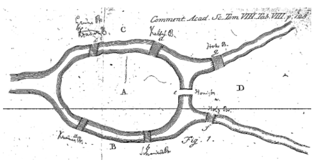
1 Background

The Königsberg bridges problem (KBP) is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736 laid the foundations of graph theory and prefigured the idea of topology. The Königsberg bridges problem shows the beauty of mathematics to transform the incomprehensible to the obvious.

This article describes the KBP and poposes some extensions of the same.

2 The problem

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands Kneiphof and Lomse which were connected to each other, or to the two mainland portions of the city, by seven bridges.



Königsburg Bridges

The KBP may be stated as: *Is there a walk through the city that would cross each of those bridges at least once and only once. Notice that there is no such limit on how many times you visit a landmass (except those implied by the terminal points of ech edge).*

This apparently innocent looking problem, gave rise to one of the most useful branches of mathematics called "graph theory", thanks to the genius of Leonhard Euler (15 April 1707 – 18 September 1783) [1] . Graph theory finds many uses in a wide range of areas like social science, electronics, computer science and operations research.

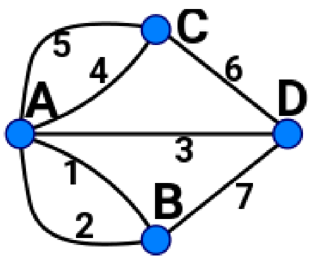


Fig. 1: Graph model

Euler used an abstract model for describing this problem, a "graph" [2]. One replaces each land mass with an abstract "vertex" or node, and each bridge with an abstract connection, an "edge", which only serves to record which pair of vertices (land masses) is connected by that bridge. A lucid tutorial on the KBP is available at [3]. It also gives an insight into the mind of the genius Leonhard Euler[3].

In graph theory, an Eulerian trail (or Eulerian path) is a trail in a finite graph that visits every edge exactly once (allowing for revisiting vertices). The original problem reduces to one of finding an Eulerean path

An Eulerian path is a walk that uses every edge of a graph exactly once. An Eulerian circuit (also called an Eulerian cycle or an Euler tour) is a closed walk that uses every edge exactly once. An extended form of the problem asks for a path that traverses all bridges and also has the same starting and ending point. Such a walk is called an Eulerian circuit or an Euler tour. Such a circuit exists if, and only if, the graph is connected, and there are no nodes of odd degree at all. All Eulerian circuits are also Eulerian paths, but not all Eulerian paths are Eulerian circuits.

Euler observed that (except at the endpoints of the walk), whenever one enters a vertex by a bridge, one leaves the vertex by a bridge. In other words, during any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it. Now, if every bridge has been traversed exactly once, it follows that, for each land mass (except for the ones chosen for the start and finish), the number of bridges touching that land mass must be even (half of them, in the particular traversal, will be traversed "toward" the landmass; the other half, "away" from it). However, all four of the land masses in the original problem are touched by an odd number of bridges (one is touched by 5 bridges, and each of the other three is touched by 3). Since, at most, two land masses can serve as the endpoints of a walk, the proposition of a walk traversing each bridge once leads to a contradiction. Since the graph corresponding to the Königsberg bridges problem has four nodes of odd degree, it cannot have an Eulerian path (and hence an Eulerian circuit).

3 Euler's Palace

We will now visit a palace, aptly named as Euler's Palace. In fact, it is a 2BHK Palace inspired from the KBP mentioned earlier. The goal is to walk through all the doors, crossing each one, once and only once. The number of times you visit a room is immaterial. Is such a walk possible ?

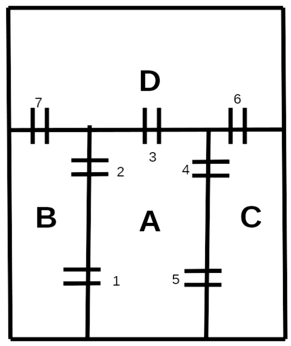


Fig. 2: Euler's Palace

4 Anti-climax

Two of the seven original bridges did not survive the bombing of Königsberg in World War II. Two others were later demolished and replaced by a modern highway. The three other bridges remain, although only two of them are from Euler's time (one was rebuilt in 1935). Thus, as of 2000, five bridges exist at the same sites that were involved in Euler's problem.

5 Post scriptum

Two questions logically extend the Koenigsburg bridges problem (KBP) [1] :

1. What is the minimum number of new bridges you would have to build, if you want a solution to the KBP problem ? Which ones ?
2. What is the minimum number of bridges you would have to destroy, if you want a solution to the KBP problem ? Which ones ?

[1]Ask the author of this article (drpartha@gmail.com) for a solution.

References

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