The most beautiful formulae / theorems / identities in mathematics

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The chief forms of beauty are order, symmetry, and definiteness, which the mathematical sciences demonstrate in a special degree.
– Aristotle

"A thing of beauty is a joy forever", which is why we all admire mathematics. Mathematicians sometimes say they find beauty (not just the visual kind) in mathematical expressions and formulas. A new brain scan study confirms that mathematical equations can activate the brain in much the same way that great art does[1].

Here are some examples :

1 A thing of beauty

This is my personal collection of formulae/theorems/identities which I consider lovely. By "lovely", I mean objects which possess a certain degree of Elegance and Simplicity (implies "easy to remember"). The formulae/theorems given below are listed in no particular order.

1. **Euler's identity**

   In 1988 The Mathematical Intelligencer, a quarterly mathematics journal, carried out a poll to find the most beautiful theorem in mathematics. Twenty-four theorems were listed and readers were invited to award each a 'score for beauty'. While there were many worthy competitors, the winner was 'Euler's equation'. Euler's identity (aka Euler's equation) states:

$$e^{i\pi} + 1 = 0 \tag{1}$$

And is often considered as the most beautiful formula/identity in mathematics.

The power of the above equation lies in the fact that it combines five important mathematical constants into one simple relationship.

2. **Euler again !**

   There is no end to the discovery of Euler. There's probably not even a single field in mathematics in which Euler didn't contribute something substantial, expressed in a very simple form. It is not surprising to find Euler in "The Second Most Beautiful Equation in the World of Mathematics " [4] :

   For any convex polyhedron, the number of vertices minus the number of edges plus the number of faces is always equal to two.

$$V - E + F = 2 \tag{2}$$

   Vertices - edges + faces = 2, this is what we know as Euler's characteristic (also known as a topological invariant).

   We can verify the above, using the five Platonic solids : Tetrahedron, Cube, Octahedron, Dodecahedron, Icosahedron.

$$\begin{aligned} Euler's\ characteristic : V - E + F &= 2 \qquad (3)\\ Tetrahedron : 4 + 4 - 6 &= 2 \qquad (4)\\ Cube : 6 + 8 - 12 &= 2 \qquad (5)\\ Octahedron : 8 + 6 - 12 &= 2 \qquad (6)\\ Dodecahedron : 12 + 20 - 30 &= 2 \qquad (7)\\ Icosahedron : 20 + 12 - 30 &= 2 \qquad (8)\end{aligned}$$

3. **Pythagoras' theorem**

   The most popular and fascinating theorem in Euclidean geometry takes a premium position in the list.

   If AB, BC and AC are three sides of a right angled triangle ABC, where AC is the hypotenuse,

$$AC^2 = AB^2 + BC^2 \tag{9}$$

4. **Heron's formula**

   The area A of a triangle, whose sides are a, b, c, is given by :

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{10}$$

   where:

$$s = \frac{a+b+c}{2}$$

5. **Bayes theorem**

$$P(A|B) * P(B) = P(B|A) * P(A) \tag{11}$$

   Or,

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)} \tag{12}$$

6. **Log magic**

$$log(1+2+3) = log(1) + log(2) + log(3) \tag{13}$$

7. **Cayley - Hamilton theorem**

   every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

8. **Euclid's algorithm**

$$If\ a\ and\ b\ are\ integers\ and\ a > b,\ then\ gcd(a,b) = gcd(a(modb), b) \tag{14}$$

9. **Pascal's triangle**

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

This amazingly simple and beautiful geometric arrangement of numbers (integers), reveals several mathematical concepts e.g. coefficients of a binomial expansion, combinations (choosing), Fibonacci numbers.....

10. **Cauchy-Riemann Equations**

    The Cauchy-Riemann Equations [2] form a necessary and sufficient condition for a complex function to be complex differentiable, that is, holomorphic

    Let
    $f(x, y) = u(x, y) + iv(x, y)$

    where
    $z = x + iy$

    so
    $dz = dx + idy$

    If f is complex differentiable, then the value of the derivative must be the same for a given dz, regardless of its orientation.

    $(\partial u)/(\partial x) = (\partial v)/(\partial y)$

    and

    $(\partial v)/(\partial x) = -(\partial u)/(\partial y)$

11. **Trigonometric gem #1**

    If A, B, C are vertices of a triangle, and sides a,b,c are a = BC, b = CA, c = AB

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)} \tag{15}$$

12. **Trigonometric gem #2**

$$sin(x - y)sin(x + y) = (sin(x) - sin(y))(sin(x) + sin(y)) \tag{16}$$

13. **Trigonometric gem #3**

    X+Y+Z = X*Y*Z if
    X = tan(A)
    Y = tan(B)
    Z = tan(C)

$$and\ A + B + C = \pi \tag{17}$$

14. **Fermat's last theorem**

    There is an intriguing story why this is called as Fermat's "last" theorem. It is strikingly similar to Pythaogoras' theorem, and it challenged mathematicians for more than a century before it could be proved mathematically. The theorem can be stated as follows:

    > *There do not exist four positive integers, the last being greater than two, such that the sum of the first two, each raised to the power of the fourth, equals the third raised to that same power.*

    In case you are trying to figure out what that means, take a look at a mathematical version of the same statement:

    > There do not exist positive integers x, y, z, and n, with $n > 2$, such that $x^n + y^n = z^n$.

    Which version is clearer and less confusing ? You be the judge. Now, you know why people say "maths is beautiful".

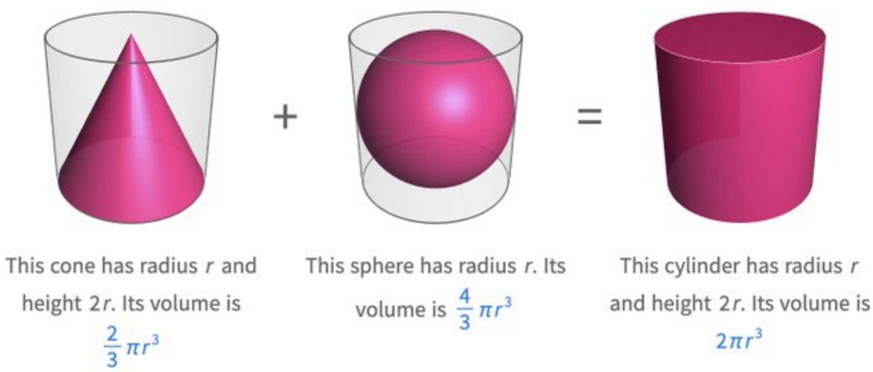
15. **The gem from Archimedes**

    Archimedes discovered that if you add the volume of a cone and a sphere, you get the volume of their bounding cylinder. This is one of the most beautiful results in 3D geometry!

    Cone-volume + sphere-volume = cylinder-volume

    $(2/3) * \pi * r^3 + (4/3) * \pi * r^3 = (2 * \pi * r^3)$

    See:



16. **A parting gift**

    Found in a schoolboy's scrap book :

$$6 + 9 + (6 * 9) = 69$$

    This remarkably simple equation uses just two digits (6,9) and just two arithmetic operators (*,+).

That's not all, read on to know more.

2 The ugly duckling

Just to contrast with the examples above, take a look at the following expression :

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 (396^{4k})} \tag{18}$$

This expression, a creation of Ramanujan, is reputed to be the ugliest equation in mathematics [1] .

But some people find it remarkable, because Ramanujan's series for π converges extraordinarily rapidly (exponentially) and forms the basis of some of the fastest algorithms currently used to calculate π .

Compare this with the more frightening formula, known as Chudnovsky algorithm [3], which gives a popular and fast method for calculating the digits of π

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(545140134k + 13591409)}{(3k)!(k!)^3 (640320)^{3k+3/2}}$$

2.1 What makes the maths teacher wild

I learnt these gems from a schoolboy:

$$\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4} \tag{19}$$

The answer was right, but the teacher was not very happy to see this !

But wait, until you see some more masterpieces to make your teacher go really mad !

$$\frac{\sqrt{\cancel{2}}}{\cancel{2}} = \sqrt{}$$

$$\frac{64}{16} = \frac{\cancel{6}4}{1\cancel{6}} = \frac{4}{1} \tag{20}$$

$$\frac{26}{65} = \frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5} \tag{21}$$

$$\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5} \tag{22}$$

$$\frac{49}{98} = \frac{4\cancel{9}}{\cancel{9}8} = \frac{1}{2} \tag{23}$$

$$\frac{166}{664} = \frac{1\cancel{6}\cancel{6}}{\cancel{6}\cancel{6}4} = \frac{1}{4} \tag{24}$$

This proves how easy it is to be innovative (and wrong) with mathematics.

3 Au revoir

Of course, this list is undeniably incomplete. There will be more entries, as I discover more gems.

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References

[1] David Freeman,
Here's Proof That Beautiful Math Equations Affect The Brain Just Like Great Art
https://www.huffingtonpost.in/entry/beautiful-math-equations-brain-great-art_n_4789667

[2] Weisstein, Eric W,
Cauchy-Riemann Equations.
http://mathworld.wolfram.com/Cauchy-RiemannEquations.html

[3] Wikipedia, Chudnovsky algorithm,
https://en.wikipedia.org/wiki/Chudnovsky_algorithm

[4] Samrat Dutta, The Second Most Beautiful Equation in the World of Mathematics
https://www.cantorsparadise.com/the-second-most-beautiful-equation-in-the-world-of-mathematics-4acbe761f637

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