Conditional statements in symbolic logic [1]

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1 Introduction

This document explains the role and usefulness of logical conditional statements (aka logical implications) used in symbolic logic. This document is part of the non-traditional teaching initiatives of the author. You can find more details (and useful learning material) at [4]:

2 Symbolic logic

Symbolic logic is the science of studying logical statements (propositions), in an abstract sense, using symbols to replace the statements, and logical connectives (aka logical operators) to create complex logical expressions. This kind of abstraction, helps us to avoid ambiguities caused by natural language expressions. A classic example, which the author often likes to state, is as follows.

Consider the statements:

1. Ramu eats food with his left hand
2. Ramu plays football with his wife

The second statement means that Ramu and his wife play football together. Does this mean that Ramu eats food and his left hand together (like eating fish and chips) ?

The first statement could mean that Ramu uses his left hand to eat food. So, can we say that Ramu uses his wife to play football (to kick her around the field) ? Imagine the semantic confusion which would prevail when we

[1] Texts shown in wine-red colour are click-sensitive hyperlinks.

combine the two statements like this : *Ramu eats food with his left hand, and plays football with his wife*.

Let us take another example. The negation of *Jack and Jill went up the hill* would be – *Jack and Jill did not go up the hill* . Does this mean :

1. Neither Jack nor Jill went up the hill
2. Jack went up the hill but Jill did not
3. Jill went up the hill but Jack did not

Natural language expressions are full of such semantic traps. Symbolic logic, helps us to express complex statements in an unambiguous fashion. Having made symbolic expressions, it is also possible to manipulate them, and derive and prove, useful inferences and deductions. Such inferences may not be explicitly visible in the expressions we started with. Symbolic logic is the fundamental tool used in automatic reasoning systems. The "conditional" statement (implication) described in this document, is an essential component of symbolic reasoning.

In logical statements (propositions) as follows, we use upper case We letters use the e.g. symbols P, Q to denote logical propositions.

$\vee$ (Boolean OR, disjunction)

$\wedge$ (Boolean AND, conjunction)

$\neg$ (Boolean NOT, negation)

to denote the corresponding logical operations on propositions. We will also study the conditional (implication) operator denoted by $\longrightarrow$.

The $\vee$, $\wedge$, $\longrightarrow$ are binary operators.
The operator $\neg$ is a unary operator.

This document will spend most of the time, examining the implication operator $\longrightarrow$, since it is of great importance in symbolic logic.

3 The conditional

3.1 Definition

The conditional statement is an essential component of logical reasoning. It is used in a wide variety of contexts.

P	Q	P $\longrightarrow$ Q
T	T	T
T	F	F
F	T	T
F	F	T

Table 1: *Truth table for the conditional*

Let P and Q be two logical statements (propositions). The statement ($P \longrightarrow Q$) is called a conditional statement (or implication statement). It is a logical statement and can therefore take a TRUE (denoted by T) value or a FALSE (denoted by F) value (as given by the last column of the truth table shown on the left). The proposition P is called the premise (or antecedent), and the proposition Q is called the conclusion (or consequent). The conditional statement (the conditional implication statement) P $\longrightarrow$ Q is a short (and unambiguous) form of saying any of the following:

1. antecedent $\longrightarrow$ consequent
2. premise $\longrightarrow$ conclusion
3. if P then Q
4. P implies Q
5. P is sufficient for Q
6. Q is necessary for P

An example : Let us take an example. Let N be an integer. Let P and Q denote respectively :

P : N is a prime number
Q : N is odd and is greater than 2

We now have the implication P $\longrightarrow$ Q, which means that:

- If N is a prime number, then N is odd and greater than 2
- "N is a prime number" implies that "N is odd and is greater than 2"

- It is sufficient for N to be a prime number, so that N is odd and greater than 2
- It is necessary that N be an odd number greater than 2, so that N is prime

Exercise :

1. Try to fit the above example (P $\longrightarrow$ Q) in a truth table. Take different values for N, and verify that the truth table actually conforms to the implication statement P $\longrightarrow$ Q.
2. Confirm that the implication statement $P \longrightarrow Q$ is equivalent to the logical expression $(P \wedge Q) \vee (\neg P)$
   Hint : Use a truth table.

This simple idea can now be used in more complex logical arguments. We will see how. For all the following subsections, we assume a statement S, of the form : $S : P \longrightarrow Q$

3.2 Converse

Let us assume a statement S, of the form : $S = (P \longrightarrow Q)$. If S_{con} is $(Q \longrightarrow P)$, S_{con} is said to be the converse of S (and vice-versa).

The converse of the statement made in Section 3.1 is : If a number is odd and greater than 2, it is prime. This statement is obviously not true (e.g. N=15) As shown above, S and S_{con} need NOT have the same truth value – S could be true while S_{con} is false, and S_{con} could be true while S is false. In other words, $P \longrightarrow Q$ but $Q \nrightarrow P$.

We will illustrate this using the example given in 3.1. For instance, all prime numbers are odd (and greater than 2), but all odd numbers greater than 2 are not prime (e.g. 15).

This takes us to a special form of converse – the biconditional.

3.3 Biconditional

There are special situations where $\{P \longrightarrow Q\} \longrightarrow \{Q \longrightarrow \text{P}\}$ This condition is denoted by P $\longleftrightarrow$ Q.

Example : Let ABC denote an equilateral triangle. Let us define two propositions P and Q, as follows:

P : AB = BC = CA
Q : $\angle A = \angle B = \angle C$

By the theorems of Euclidean geometry we can prove that:
(P $\longrightarrow$ Q) $\wedge$ (Q $\longrightarrow$ P) i.e. P $\longleftrightarrow$ Q
P is a necessary and sufficient condition for Q (and vice-versa).

3.4 Contrapositive

The contrapositive of $S : (P \longrightarrow Q)$ is the proposition : $(\neg Q \longrightarrow \neg P)$.

Exercise : Prove that the contrapositive of S is equivalent to S

Example : We can state the contrapositive of the statement given in the example in Section 3.1 as:
If a number is not odd and greater than 2, it is not a prime number.

3.5 Inverse

The inverse of S is the proposition : $\neg P \longrightarrow \neg Q$

Exercise : Prove that the inverse of S is equivalent to the converse of S

Example : We can state the inverse of the statement given in the example in Section 3.1 as: If a number is not prime, it is not odd and greater than 2. This is obvisusly not true, like the converse (e.g. N=15).

3.6 Modus ponen, modus tollen

The "conditional" described above has two useful applications in logical reasoning and proof [5]:

1. Modus ponen (method of affirmation)
2. Modus tollen (method of negation)

Modus ponen : Using the conditional, we can build an argument of the kind:
$P \wedge (P \longrightarrow Q) \longrightarrow Q$.

Modus ponen (if P is true, Q is true; but P is true; therefore, Q is true)

Exercise : Prove that this logical argument is a tautology i.e. its truth value is always true, for all truth values of P and Q.

Modus tollen : Using the conditional, we can build an argument of the kind:
$\neg Q \wedge (P \longrightarrow Q) \longrightarrow \neg P$
Modus tollen (if P is true, Q is true; but Q is false; therefore P is false)

Exercise : Prove that this logical argument is a tautology i.e. its truth value is always true, for all truth values of P and Q.

Example : Let us take an example. Let N be an integer. Let $(P \longrightarrow Q)$. Let P and Q denote respectively :

P : N is a prime number
Q : N is odd and is greater than 2

Modus ponen : P is true for N=7 , and $(P \longrightarrow Q)$, so Q is true
So N is odd and greater than 2.

Modus tollen : Q is false for N=12 , and $(P \longrightarrow Q)$, so P is false
So N is not a prime number.

From the above, we see how the conditional statement is used in symbolic logic [5]. The above discussion also demonstrates how to avoid the traps caused by semantic ambiguities in reasoning with natural languages.

4 Conclusions

We have seen some features of the "conditional" statement, used in symbolic logic. At each point highlighted in the text, we have tried to include some self-evident examples. This document is also an effort to demonstrate the power of LaTeX for producing mathematically-rich text with logic symbols. This article [6] and similar articles are downloadable from the w-w-web [2]. If you have any questions on the above, you can always ask the author by email, at drpartha@gmail.com as explained in [1] .

5 Closing remarks

The LaTeX source can be obtained by sending an email to drpartha@gmail.com. This article is released under a liberal license [3] and published on the web at [6]. Please mention the Reference Code, and Version code, given at the top of this document. Please follow the "basic rules of decency" explained in [1]

Readers are requested to send all remarks, criticisms, queries and suggestions to the author at drpartha@gmail.com.

6 About the author

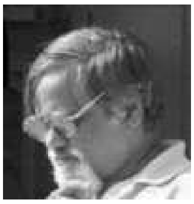

Figure 1: The pensive Professor

Parthasarathy is an aggressive supporter and a loyal practitioner of FOSS. He teaches discrete mathematics, and preaches LaTeX and Linux, to students of Computer Science. His tutorial material is downloadable from the web, under a liberal license (CC-BY-SA). He would be happy to assist anyone, particularly students, teachers, and institutions, who are genuinely interested in these topics.
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