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A closer look at Permutations and Combinations [1]

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Abstract

Combinatorics is a branch of mathematics concerning the study of finite or countable discrete structures [1]. Combinatorics is based on two concepts: permutations and combinations. Combinatorial problems arise in many areas of pure mathematics, and applied mathematics. This report is a study of a particular aspect of permutations and combinations. This subject will be explored using various examples from real life.

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[2]Author's profile : `http://drpartha.org.in/profpartha/profile.pdf`

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1 Introduction

Combinatorics is a branch of mathematics concerning the study of finite or countable discrete structures [1]. Combinatorial problems arise in many areas of pure mathematics, and applied mathematics. This report, inspired by [2], is a study of a particular problem in combinatorics, that of permutations and combinations (of a particular kind).

2 Permutations and combinations

Let us revise some fundamentals of combinatorics. The fundamental concepts in combinatorics are :

- Combinations
- Permutations

Warning : the terms "permutation", and "combination" are badly chosen terms. They would more aptly be called as "arrangements", "choice", or "selection". The term is used in several confusing ways. There is also a major confusion regarding the symbols to be used for these.

A **combination** of r objects out of n objects (a selection/choice of r objects out of n objects) is denoted by any one of the various symbols :: ${}_n\mathrm{C}_r$ OR C(n,r) OR $\binom{n}{r}$ OR C_r^n OR ${}^nC_r$. It may be seen as selecting r objects from an input bin containing n objects, and then depositing them in some output bin.

Technically speaking, ${}^n\mathrm{C}_r$ is the cardinality of the set of all selections of r objects out of n objects. ${}^n\mathrm{C}_r$ is therefore a positive integer.

- Since we are talking of a set of selections, we do not bother about the order in which the selections are made.
- It is implicitly assumed and accepted that none of the r objects you choose is repeated within each selection.
- Since each selection is itself a set, we do not bother about the order in which the objects are arranged inside each selection .

The case of multiple copies i.e. *repetition* of the an object is considered as a *selection with replacement* . Some authors use the term "replacement" to denote “repetition”. Selection with replacement / repetition is similar to selecting one out of r objects from an input bin containing n objects, and then depositing it in some output bin, and also replacing the selected object back into the input bin. This aspect is discussed in more detail in the next section 2.1

A **permutation** is the total number of ways of arranging r objects, chosen out of a collection of n objects. It is denoted by : ${}^{n}P_{r}$. This is the same as selecting r objects out of n (in any order), and then arranging these r objects (out of r objects) in all possible sequences.

We will use ${}^{n}P_{r}$ for a “permutation of r objects out of n objects” (arrangements of r objects out of n objects).

It is easy to work out an expression to compute ${}^{n}P_{r}$, given n and r.To choose r objects out of n objects, we can perform the following operations:

The first object can be any one of the n objects available. This gives us n possibilities for this object.

The second object can be any one of the $n-1$ objects available. This gives us $n-1$ possibilities for this object.

The third object can be any one of the $n-2$ objects available. This gives us $n-2$ possibilities for this object.

$\vdots$

The rth object can be any one of the $n-r+1$ objects available. This gives us $n-r+1$ possibilities for this object.

For selection of all the r objects put together, we get the following total possibilities:

$$
\begin{aligned}
{}^{n}P_{r} &= n*(n-1)*(n-2)\dots(n-r+1) && (1)\\
&= \frac{n*(n-1)*(n-2)\dots(n-r+1)*[\ (n-r)*(n-r-1)*\dots 1]}{[\ (n-r)*(n-r-1)*\dots 1]} && (2)\\
&= n!/(n-r)! && (3)
\end{aligned}
$$

It is a trivial matter to prove that ${}^{r}P_{r}$ = r ! (since $0! = 1$)

If the order of objects in a given selection doesn't matter then we have a combination (several selections, each with a different ordering of objects selected, can be considered to be the same and merged into one). If the order does matter, we must count each selection/choice separately, which gives us a permutation. One could say that a permutation is a set of combinations (without shuffling of objects).

$$
\begin{aligned}
{}^nP_r &= {}^nC_r.\ {}^rP_r && (4)\\
&= {}^nC_r.r! && (5)\\
{}^nC_r &= \frac{{}^nP_r}{r!} && (6)\\
\therefore {}^nC_r &= \frac{n!}{r!.(n-r)!} && (7)\\
\because {}^nP_r &= \frac{n!}{(n-r)!} && (8)
\end{aligned}
$$

Now, let us see what ${}^nC_{(n-r)}$ is :

$$
\begin{aligned}
{}^nC_{(n-r)} &= \frac{n!}{(n-r)!.(n-(n-r))!} && (9)\\
&= \frac{n!}{(n-r)!.(n-n+r)!} && (10)\\
&= \frac{n!}{(n-r)!.r!} && (11)\\
&= {}^nC_r && (12)\\
\therefore {}^nC_{(n-r)} &= {}^nC_r && (13)
\end{aligned}
$$

This confirms the fact that, choosing r objects out of n objects, is the same as choosing $(n-r)$ objects to reject out of n objects.

That works fine, as long as the r objects are distinct i.e. there is no repetition of objects (each object is available for selection only once).

Let us examine the case when this is not true. Intuitively, we can imagine that as soon as an object is chosen from a bin, it is put back into the original bin so that it may be selected again. Thus we would have replaced $(r-1)$ objects when the last of the r objects has been selected. This is equivalent to choosing r objects out of $n+(r-1)$ objects. Each selection is now a multiset.

2.1 Combinations with repeated objects

The notation $((C^n_r))$, proposed by Stanley [5] denotes the choice of r objects out of n ojects with repetition, to distinguish it from C^n_r the choice of r objects out of n objects without any repetition.

In the case where the choice permits repetition of the objects chosen, we use [5]

$$((C\,^n_r)) = \left(C\,^{n+r-1}_r\right) \tag{14}$$

By using the same logic as in equation 13 above, we get:

$$((C\,^n_r)) = \left(C\,^{n+r-1}_{n-1}\right) \tag{15}$$

Why is this aspect of combinatorics interesting ? Let us study some simple examples, to illustrate the above principles:

3 Examples

3.1 Child in a candy store

This example will illustrate the role of "repetition" in computing combinations, and will serve as the basis of our discussions on combinations.

A child walks into a candy store (with his dad) and is allowed to buy 2 pieces of candy. The store has chocolate (C), gummies (G), and horrible Chinese candy (H). The dad also imposes that the child can not buy two (or more) candies of the same type. The obedient child has the following options only : CG, CH, HG. This is equivalent to

$$^3C_2 = 3$$

Now, the dad is less harsh and generously lets the child buy any two candies (even if they are of the same type). The excited child has the following options now: CG, GH, CH, CC, GG, HH. This is equivalent to

$$((C^{3+2-1}_{3-1})) = {}^4C_2 = \frac{4!}{2!(4-2)!} = 6 \tag{16}$$

Imagine the special day (the child's birthday) when dad takes his child to a bigger store (they have 100 types of candies) and permits the child to buy a handfull of candies (let us say 7). Enumerating all the possible options would obviously be very difficult. Equation 11 and equation 14 can make life easy for all of us (but they still involve very big numbers).

Having understood the difference between “selection without repetition” and “selection with repetition”, we will explore some examples for adults (who do not crave for candies).

3.2 Numeric codes

Imagine a numeric code which can have 4 digits in a specific order, the digits are between 0-9. Such codes are commonly used as PIN for debit cards/ credit cards, or for OTP passwords in mobile phone based applications.

Case#1 : How many 4 digit codes are possible with the digits 0 . . . 9 ?

The first digit of four-digit code can be any of the ten numerals $0 \ldots 9$. This gives 10 ways of choosing the first digit. Similarly, there are ten ways to choose each of the next three digits. The total number of possibilites would be :

$$10 * 10 * 10 * 10 = \tag{17}$$
$$= 10^4 \tag{18}$$
$$= 10000 \tag{19}$$

Looking at this in another way, a four digit code could be anything between 0000 to 9999, hence there are 10,000 combinations in all.

This case covers :

- Selection with repetition : Every digit could be used more than one time within a code. Thus, 1729, 1779, 1722 . . . are all valid
- Permutation allowed : The codes can have the same digits repeated between them. Thus 1729, 1279, 1972 . . . are all valid and are counted separately, otherwise they are considered to be the same (and counted as one) .

Incidentally, 1729 is a famous number related to the Indian mathematician Ramanujan [3].

Case #2 : How many unique 4 dgit codes are possible ?

In this example the following conditions hold:

- Selection without repetition : No digit could be used more than one time within a code. Thus, 1229, 1779, 1722 . . . are not valid
- Permutation not allowed : No two code can have the same digits repeated. Thus 1729, 1279, 1972 . . . considered the same, and are not counted separately.

The number of combinations of n objects taken r at a time is determined by the following formula:

$$
\begin{aligned}
C(n, r) &= n!/(n-r)!r! && (20)\\
&= 10!/6!4! && (21)\\
&= 210 && (22)
\end{aligned}
$$

How many 4 dgit codes are there if one digit may only be used once ?

Since we are told in the question that a digit may be used only once, it limits our number of combinations. Thus, if 1729 is an acceptable code, 1179, 1779, 1722 . . . are not valid codes.

In order to determine the correct number of permutations we simply plug in our values into our formula:

$P(n, r) = 10!(10-4)! = 10.9.8.7.6.5.4.3.2.1/6.5.4.3.2.1 = 5040$

3.3 Piggy banks

Imagine having n coins and m piggy banks. Let $m > n$ In how many ways can all the coins be filled in the piggy banks ?

Specifically, assume we have four coins and three piggy banks. How can the coins be kept in the piggy banks ?

The solutions may be represented by the following figure:

$| * | * * * |0|$
$| * *| * *|0|$
$| * | * | * *|$

In the above sketch :

- Each piggy bank is the space between to adjacent |
- A $*$ denotes a coin
- A 0 symbol denotes no coins

3.4 Morse codes

Morse code is a method of transmitting text information as a series of on-off tones, lights, or clicks that can be directly understood by a skilled listener or observer without special equipment. It is named for Samuel F. B. Morse, an inventor of the telegraph (in around 1837). The International Morse Code [4] encodes the ISO basic Latin alphabet, some extra Latin letters, the Arabic numerals and a small set of punctuation and procedural signals (prosigns, queries) using standardized sequences of symbols called “dots” and “dashes” [6] . Each character, encoded in Morse code, can contain one or more of the two Morse symbols “dot” or “dash” .

In addition, Morse code uses pauses (silence/gap of different durations), to separate

- consecutive symbols used in a character
- consecutive characters in a word
- consecutive words

The prosigns, queries and abbreviations are combinations of two or more characters sent together with no pause / gap in between.

The pause/gap between the components of one character is one unit, between characters is three units and between words seven units. A pause (silence/gap) is interpreted appropriately by the person reading the encoded message.

4 Concluding remarks

This article was typeset in LaTeX using Kile, under a Linux system. The LaTeX source file of this article can be obtained by sending a request to the author drpartha@gmail.com. Please mention the Ref. code and the Ver. code given near the top of this report.

As always, constructive suggestions, comments and remarks may be sent to the author.

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