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# Using graph colouring for a class of resource scheduling problems - a tutorial

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# A tutorial on graph colouring for a class of resource scheduling problems

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# Graph colouring for a class of resource scheduling problems - a tutorial

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**Abstract**

This article[1] presents the application of graph colouring for, resource sharing which involves conflict. A conflict is defined as a situation which necesssitates mutual exclusion. Such problems occur commonly in our daily life. Graph colouring helps the user to approach the problem in a systematic and mathematically sound fashion. A simple example is used for illustrating the approach.

## 1 Introduction

Resource sharing or resource scheduling is a common and useful application encountered in daily life. Resource sharing (or resource scheduling) is used in a wide variety of domains. Some notable examples are :: roster management, allocation of frequencies in cell phone networks, tank-farm management, examination scheduling, multi-colour maps and atlases, etc. This report presents a mathematically sound approach for solving problems in resource sharing.

Graph colouring started out from an innocent observation from creating multi-coloured atlases (maps). It turns out that graph colouring has many interesting and challenging aspects. The necessary background and related concepts can be found in [4] and [1]. In this report, we study an application of graph colouring, not related to atlases and maps.

# 2 A simple application problem

Suppose you want to schedule final exams. On a given day, exams are held concurrently (at the same time). Being very considerate, you want to avoid having a student do more than one exam a day. We shall call the courses A B C D. Let us imagine there are five students, say P Q R S T, taking some of these courses, and appearing for the final exam.

The question is, what is the minimum number of days needed to conduct all the exams ?

**Note :** The terms: contention matrix, conflict matrix, and conflict graph (defined below) have been introduced by the author, and are not defined in the standard literature..

## 2.1 Case #1

Imagine a class of five students - P Q R S T, and courses/subjects A B C D.
Student P has registered for A and B
Student Q has registered for C and B
Student R has registered for C and D
Student S has registered for A and D
Student T has registered for C and B

# 3 Contention matrix

The following matrix (table) lists the students and the subjects they have registered for. A * indicates that the particular student (indicated by the row) has registered for the particular subject (indicated by the column). For example, student S has registered for subjects A and D.

| | A | B | C | D |
|---|---|---|---|---|
| P | * | * | | |
| Q | | * | * | |
| R | | | * | * |
| S | * | | | * |
| T | | * | * | |

# 4 Conflict matrix

The above matrix (table) indicates potential conflicts. For example, the examination for A and B cannot be held on the same day, because student P will suffer. He cannot appear in two examinations at the same time. We will use the above matrix, and prepare a matrix (table) which will show potential conflicts and clashes.

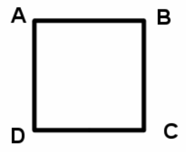
| | A | B | C | D |
|---|---|---|---|---|
| A | | * | | * |
| B | * | | * | |
| C | | * | | * |
| D | * | | * | |

A * in position ij indicates that there is a clash between subject i and subject j. At least one student has registered for subject i and subject j. For example, we see from the above that there is at least one student who has opted for both A and B (student P) and at least one student has opted for both A and D (student S). The matrix however does not tell who the affected students are.

Notice that the relationship defined by the above matrix is *symmetric*. i.e. if A clashes with B, B clashes with A. The relationship is *irreflexive*, i.e. A cannot clash with itself (it would be meaningless). This gives rise to a symmetric matrix. Additionally, the diagonal is blank. The relationship is not transitive. If A is in conflict with B, and B is in conflict with C, it is not necessary for A to be in conflict with C.

# 5 Conflict graph

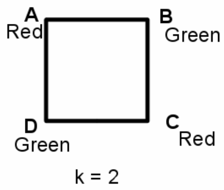
It is now an easy step to convert the above conflict matrix to a graph (conflict graph). The vertices of this graph are the items which could be in conflict. Two vertices are connected by an edge if and only if they are in conflict. We get the following conflict graph for the example we are studying.




# 6 Solution of the application problem

The next step would be to colour the vertices of the graph. Assign a "colour" to each vertex, such that no two adjacent vertices (vertices connected by an edge) get the same color. This is known as the graph colouring problem (aka graph coloring problem), and is described extensively in [1] and [4].

We get the following solution for the graph colouring of the conflict graph shown above.




We can now interpret the above colouring as follows. A and C carry the same colour (red), so they can be held on the same day. B and D (green) can be held on the same day (but other than the red day). The number of colours used is known as the *chromatic number* of the graph (k=2).

In other words, we need just two days for conducting all the four exams.

Day 1 (red) :: A and C
Day 2 (green) :: B and D

# 7 Exercises

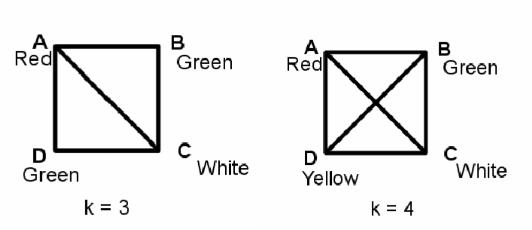
We extend the above, and re-evaluate the minimum number of days needed. We study two new cases.

1. Case #2 : Student P has registered for A B and C, Q has registered for B and C, R has registered for C and D, S has registered for A and D, T has registered for B and C

2. Case #3 : Student P has registered for A B and C, Q has registered for B and C, R has registered for B C and D, S has registered for A and D, T has registered for B and C

How would our analysis change, in view of the above ? How many days (minimum) would be required to conduct exams in all the subjects ?

For each of the cases, we give the conflict graph (with colouring). The chromatic number is given below each figure. The exam schedule is also given below for each case. We leave the detailed procedures as an exercise for the reader.
Notice that the number of subjects has not changed. The number of students has not changed either. Why does the minimum number of days needed, change ?




# 8 A word of caution

The example we have taken is admittedly very simple, and may be solved by clever reasoning and common sense. In general, real life problems tend be far more confusing. A systematic and mathematically sound approach is needed in such cases.

However, graph colouring [1] is deceptively simple. The idea of colouring a graph is very straightforward, and it seems as if it should be relatively straightforward to find a colouring. It turns out that this is an extremely difficult task. A simple algorithm for graph colouring is easy to describe, but potentially extremely expensive to run.

In terms of computational complexity it is actually NP-hard. There is no known algorithm for optimal graph colouring (for a general graph) which isn't exponential; and that further, if there were a non-exponential algorithm for it, there would be a non-exponential solution for all NP-complete problems, but a non-exponential solution to another NP-complete problem wouldn't necessarily produce a non-exponential time solution for graph colouring!

In fact, the only general solution to finding an optimal graph colouring is exhaustive search: start with one node, give it a colour, assign non-conflicting colours to its neighbors, and so on. Try it with two colours, if you get no result, then try with three, and so on. There are a lot of fancy algorithms that try to improve on that - both by reducing the search space, and by using heuristics, and by trying a parallel approach. With a combination of those techniques, we can get colourings to be quite efficient in specific cases - but if always want the optimal colouring of any graph, then there's no way (or at least, no way that anyone knows about) to always get the optimal result quickly. The search becomes prohibitively expensive as the graph size increases.

In fact, this observation reinforces the fact that resource scheduling is itself a computationally difficult problem. At best, graph colouring gives us an approach to expressing resource scheduling problems in an elegant mathematical framework.

# 9 Concluding remarks

The author of this paper [2] teaches discrete mathematics, to undergrad students of Computer Science, in India. He also runs a specialised enterprise which uses LaTeX and FOSS tools extensively (Algologic Research and Solutions). 

This paper, predictably, was made using LaTeX . It used the Kile front-end provided by Suse Linux. It uses the AMSmath package, downloaded from the CTAN site. You can get the LaTeX source of this text, as well as the rendered version (pdf file), by sending a request by email, to the author, at: drpartha@gmail.com. This paper, and a whole lot of similar tutorial material, is available on the web [3].

Please send your comments, remarks, and suggestions to the author :

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## 10 Acknowledgments

This report is the result of recent discussions with Prof. Christopher Mawata (svf234@mocs.utc.edu), Department of Mathematics, University of Tennessee at Chattanooga, USA. The author thanks Prof. Mawata sincerely for all the guidance and support.

## References

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http://www.imada.sdu.dk/ btoft/Graphcol/overview.html

[2] About the author ::
http://www.freewebs.com/profpartha/parthas.htm

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http://www.freewebs.com/profpartha/publications/downloadables.htm

[4] A first course in graph theory and combinatorics, S M Cioba, M Ram Murty, Pub.: Hindustan Book Agency, Gurgaon, India

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