Use of mathematical modelling and abstraction – a case study

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Ref.: insanity.tex
Ver. code: 20161026a

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Abstract

Instant insanity is a simple-looking puzzle, available in many toy stores. Analysing it mathematically involves usage of many mathematical concepts. This article describes the problem and explains some of the possible approaches to solve it.

1 II puzzle – The puzzle that can make you insane

"Instant Insanity" [1] is a simple-looking puzzle, available in many toy stores. This puzzle (aka II puzzle) turns to be amazingly difficult to resolve. It also serves as a case study, to demonstrate the power of mathematical reasoning through modelling and abstraction. Instant insanity is in fact, one of a family of puzzles [2]. This family, as well as a huge collection of other puzzle related material can be found in [3].

The puzzle consists of four cubes, each face of which is painted with any one of four colours. The same set of four colours is used for all the cubes. "Solving" the puzzle involves, stacking the four cubes, one over the other, such that all four colours appear one each column. One way to solve the puzzle would be through trial and error. With a little amount of luck, one can achieve the goal of the game. But, more often than not, the result would be exactly what the name says : insanity. There are thousands of ways to line up the four cubes (see [5] for a proof), so exploring them all by hand will take a while (if you do not turn insane by then)!

Sometimes, the puzzle is visualised by using an "unfolding" of the cubes. The cubes are "cut open" along some edges, and laid flat. The six faces appear as six squares, each square sharing an edge with one (or more) other edges. It is still very difficult to solve the puzzle. A smarter way would be to use some amount of mathematical reasoning through modelling and abstraction.

2 How mathematical modelling can help

2.1 Terminology

This deceptively simple-looking toy involves the usage of many mathematical concepts :: graph theory, matrix algebra, combinatorics, advanced alebgra. To get these ideas clearly, we need to define a few terms:

Configuration : The puzzle consists of four cubes, each face of which is painted with any one of four colours. We use 4 distinct colours for painting six faces of a cube. Cofiguration is the operation of assigning one of the four colours, to each face of a cube.

Orientation : The puzzle consists of four cubes. Each cube is assumed to be placed with one face facing North. The placement of other faces gets automatically fixed by this choice. Orientation is the operation of assigning the pointing direction to a given face of the cube. e.g. North face is Red

Arrangement : The puzzle consists of four cubes, stacked one over the other. Imagine each cube is numbered according to the following convention. Cube #1 is the lowest, Cube #4 is the highest. Arrangement is an ordered collection of the orientation of each cube in the stack.

There must be a way to unambiguously identify the colouring used on each cube. The first step consists of giving distinct "names" to each of the six faces of the cube. Without loss of generality, we can name the faces according to the direction in which they are pointing, say North (N), East (E), South(S), West (W). In addition, we use two directions T and C. We avoid the names “top” (T) and “bottom” (B), to avoid a conflict with B as used for blue colour.

Imagine sitting in front of a table, and facing the North. Imagine a cube lying on the table. with exactly one face facing you. This face will be the front face or south face (S). To the right (your right) is the face called east (E), to the left is the west face (W). If you go to the left of the west face, you reach the north face (N) of the cube.The face touching the table will be called the table face (T). The face exactly opposite to T, facing the ceiling will be (C).

We then choose names of each of the colour applied on each face. We use 4 colours to paint the 6 faces : Green (G), Yellow (Y), Red (R), Blue (B). Since the cube has six faces, two of the colours will be repeated in each cube. By Polya’s Theorem [4], this leads to 2226 ways of painting each cube.

Notice that the names of the six faces (N,E,S,W,C,T), and the names of the four colours (B,R,Y,G), have been chosen so as to be distinct and unambiguous.

We can imagine an axis running (vertically) through the T and C faces of the cube. We can imagine another axis running (horizontally) through the N and S faces of the cube. Any one of the six faces could be the N face of the cube. If this face is fixed, the S face is also fixed. Now, for each of the possible 6 faces (kept facing North), we could have any of the other 4 faces as the T face (kept facing the table) by rotating the cube about the NS axis. We thus get 6 x 4 = 24 possible orientations for each cube.

Imagine a cube lying on the table. Two fundamental operations are possible on this cube.

- **H90** Turn by 90° in the horizontal plane (clockwise as seen from above, about the CT axis)
- **V90** Turn by 90° in the vertical plane (clockwise as seen from front, about the NS axis)

Our puzzle consists of four cubes stacked one over the other. Each cube can take any one of the 24 orientations. Our stack can therefore take any one of the $24^4 = 331776$ arrangements ! The entire stack (all cubes together) can be rotated about a vertical axis (by 90 ° each time). So, we have 331776/4=82944 ways of looking at the stack.

The puzzle is said to be "solved" if the following conditions hold :

- In each of these views, we must see a different colour on each cube in a vertical column.
- By implication, we must see all the four colours in each column.

2.2 Matrix manipulation approach

The orientation of a cube can be described by a 2x6 matrix. For example :

Face	N	W	S	E	C	T
Colour	B	R	Y	B	Y	G

Turning the above cube by 90°(clockwise) in the horizontal plane (about the CT axis) , will give us:

Face	N	W	S	E	C	T
Initial	B	R	Y	B	Y	G
afterH90	R	Y	B	B	Y	G

Note : N W S E will get cyclically shifted. The C and T faces remain unchanged in this operation.

Turning the above cube by 90°(clockwise) in the vertical plane (after turning it horizontally as above), will give us:

Face	N	W	S	E	C	T
Initial	B	R	Y	B	Y	G
afterH90	R	Y	B	B	Y	G
afterV90	R	Y	B	G	B	Y

Note : The E C W T faces will get cyclically shifted. The N and S faces remain unchanged in this operation.

We now have a compact and unambiguous way of describing the cube and its orientation in terms of the colours.

We use the set of cubes available with the author, as an example. Imagine a random starting configuration (unsolved puzzle). We can depict the stack of cubes as :

	N	W	S	E	C	T
Cube #4	B	B	Y	Y	R	G
Cube #3	B	B	Y	Y	R	G
Cube #2	B	B	Y	Y	R	G
Cube #1	B	B	Y	Y	R	G

It is enough to perform the two fundamental operations shown earlier, on each cube, to transform the above arrangement to a solved configuration. In the solved arrangement (there could be more than one solved arrangement), we will have ::

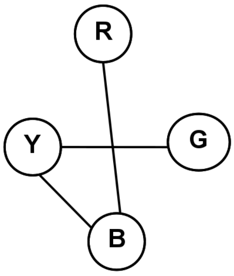
	N	W	S	E	C	T
Cube #4	R	Y	B	G	R	G
Cube #3	Y	G	G	R	R	G
Cube #2	G	B	R	Y	R	G
Cube #1	B	R	Y	B	G	R

Each row of the "solved" arrangement is a permutation of the corresponding row in the "unsolved" initial ararangement. The constraint in each row is that the relative colouring should satisfy the configuration matrix of the corresponding cube.

2.3 Graph theoretic approach

The most commonly employed tool for solving the instant insanity puzzle is based on graph theory. For the graphical solution technique, the set of cubes is represented as a single undirected "pseudograph" (aka multigraph). The term "graph" usually formally excludes more than one edge connecting any two nodes, and self loops.

Usually a cube can be modeled as a 3-regular graph, with four vertices and six edges. We can model the cubes of our II puzzle, using



Cube #1 Opposing face colours

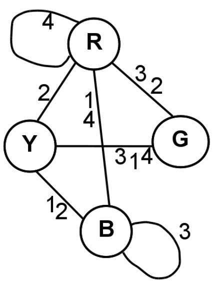
Figure 1: *Cube #1*

a special form of graphs. In classical graph theory, it is usual to describe a graph using adjacency relationships. In our case, we will use "un-adjacency" of two faces of a cube. Faces which are on opposite sides of the cube, i.e. faces which do not have any edge in common in the cube, are connected by an edge in the graph model.

A single cube, is shown in the figure on the left. We can combine all the four cubes into a single composite graph as in the figure on right. In the combined graph, the edges are labeled with numbers corresponding to the graph they represent.

We will thus have a graph with four vertices (one for each colour), and three pairs of edges (one edge for each pair of opposite faces). When the opposite faces are of the same colour, we use a self-loop as an edge.

The solution of the puzzle is reduced into a problem of finding a clique in graph, the clique should be a regular graph with degree 2. We would need to find two such cliques. One for the arrangement in the front side, and one more for back side.



Cube #1234 Opposing face colours

Figure 2: *Four cubes*

2.4 Group theoretic approach

We exploit the symmetry involved in the H90 and V90 operations, to give a group theoretic interpretation of the II puzzle. Let G be the set of orientations of a cube of the II puzzle. The set G and the operation H90 (respectively V90) form a group.

3 Concluding remarks

The approaches given above are useful in modelling the II puzzle using mathematical abstractions. There are still some limitations for the approaches described above.

- There is no easy way to derive sufficient or necessary conditions for a solution.
- There is no easy way to prove whether or not a solution exists, or how many solutions (if any) exist.

- There is no easy way to derive an algorithm which will solve the problem.

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